

Is John Conway in Heaven with the Surreal Numbers?

**The Passing of a Genius Who Saw Infinity
Raises Questions about the Nature of Mathematics**

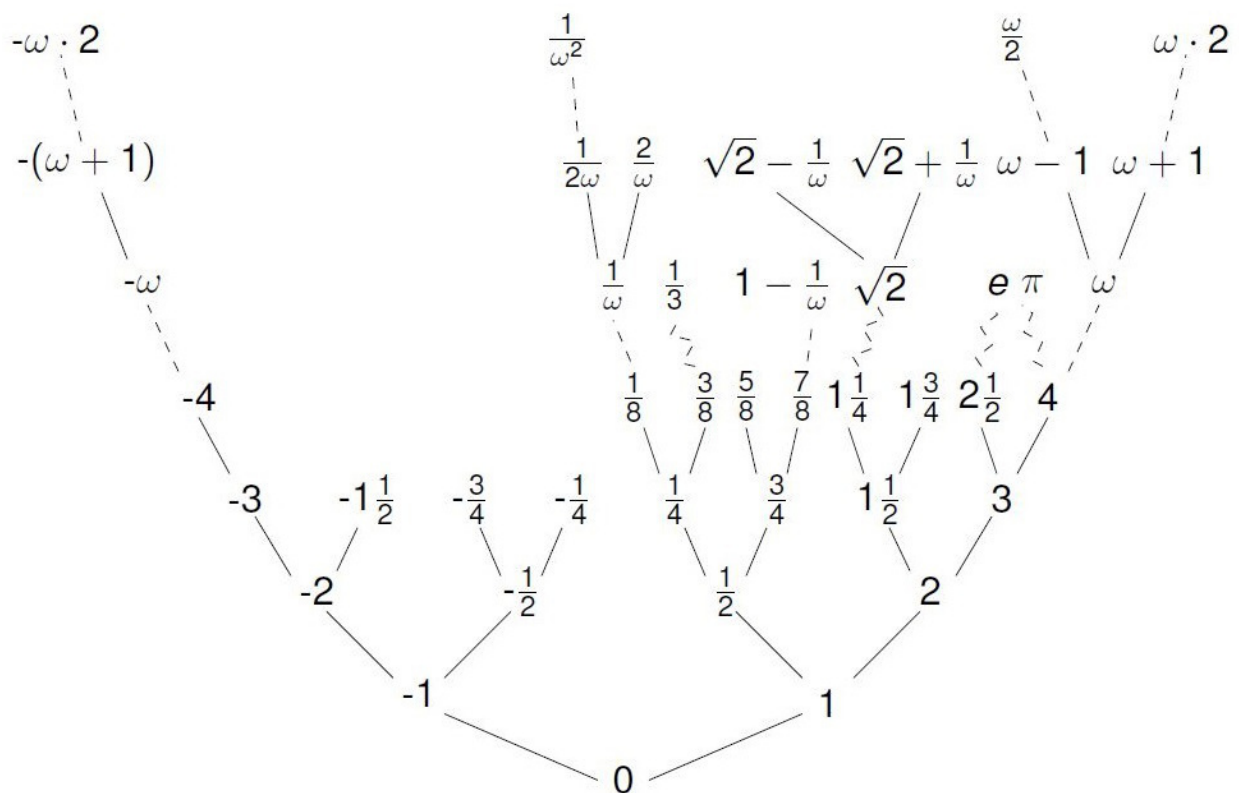
Jeffrey Ventrella, May, 2020

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With [John Conway's](#) recent passing from COVID-19, a group of friends and followers came together for a video conference to tell stories and recount memories. Included were [Donald Knuth](#), [Roger Penrose](#), [Rudy Rucker](#), [Scott Kim](#), [Ivars Peterson](#), and several other notable folks working in math, math education, puzzles, games, and art. His biographer, [Siobhan Roberts](#), and some family members, were there too.

A major topic of discussion was the system of [surreal numbers](#), which Conway discovered (or “invented”, depending on your belief system). At the risk of oversimplifying, I will just say that the surreal number system is a novel way of describing and representing all [real numbers](#), including the infinity of real numbers that snuggle in-between other real numbers, and including all [degrees of infinity](#) and the infinitesimal, using a small, elegant set of rules for generation.



(image from <https://math.illinois.edu/research/igl/projects/summer/2019/iglour-surreal-numbers>)

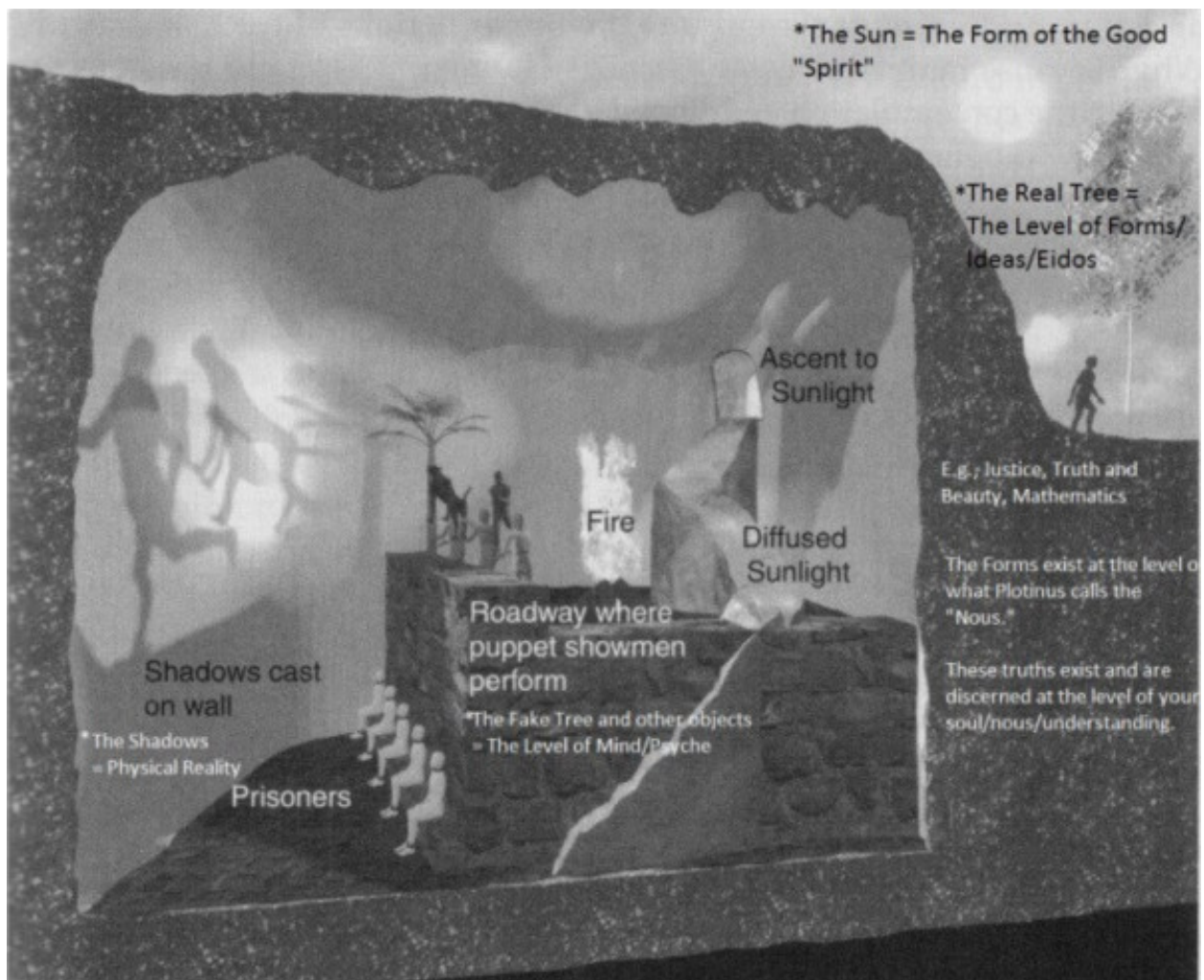
I’m not here to explain the surreal numbers. I’m here to ask questions about infinity. For a full explanation of the surreals, check out [Knuth’s book](#).

Conway considered himself a [mathematical Platonist](#): he held the metaphysical belief in the existence of eternal mathematical objects and ideas that are independent of—and *outside of*—human language and thought. It helps to be a Platonist if you’re going to say

anything authoritatively about infinitely-large numbers or infinitesimally-small numbers. This is one reason I would not be the best person to explain them.

Escaping from the cave

The allegory of [Plato's cave](#) describes a group of people who are chained to the wall of a cave for their entire lives. They face a blank wall and watch shadows projected onto the wall from objects being passed in front of a fire behind them. They give names to these shadows. The shadows represent *reality* to the prisoners.



(image from <https://voegelinview.com/platos-cave/>)

Are we mortals doomed to only see the shadows of pure, perfect mathematical entities? If so, where exactly are those entities? And who, or what, is casting their shadows?

Some people do not believe that mathematical objects and ideas are eternal and outside of human activity. In particular: infinity can be hard to swallow. (It's very big after all). But infinities and infinitesimals are essential to the surreal number system—and a lot of math besides. Can one fully appreciate the surreal number system without actually “believing” in infinity?

I would say yes; one does not have to literally *believe* in the existence of something to appreciate it, or even to use it in a practical way. However, when it comes to mathematics, *believing really helps*. In fact, it might even be necessary, on some fundamental cognitive level.

Let's get real. Infinity doesn't exist

Infinity doesn't exist other than as an abstract concept (although a powerful one, and indispensable to many branches of math). An old blog post of mine ([Very Large Numbers are not Numbers](#)) advocates the idea of an embodied math, (the origins of mathematics are based in our physical existence, and our evolved cognitive relationship with physical reality). This idea is developed in [Where Mathematics Comes From - How the Embodied Mind Brings Mathematics into Being](#), by Lakoff and Nuñez.



Another book, by [Brian Rotman](#), makes a compelling case against infinity. The book is called, [Ad Infinitum, The Ghost in Turing's Machine: Taking God Out of Mathematics and Putting the Body Back In](#). [Charles Petzold](#) gave a [review of Rotman's book](#). I will repeat a quote from Petzold that I used in my old blog post:

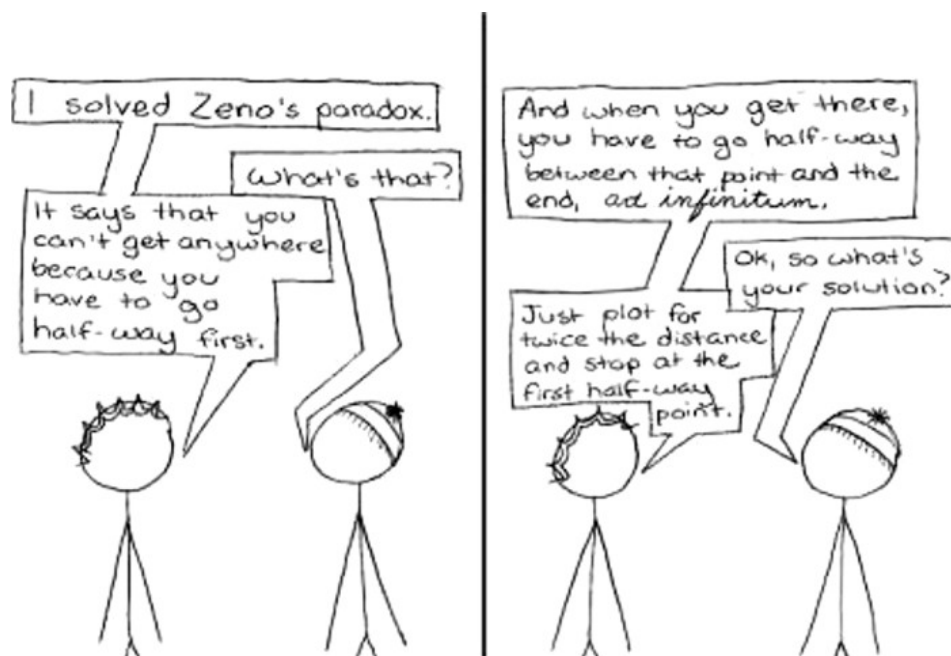
“We begin counting 1, 2, 3, and we can go on as long as we want.”

That's not true, of course. "We" simply cannot continue counting "as long as we want" because "We" (meaning "I" the author and "you" the reader) will someday die — probably in the middle of reciting a very long (but undoubtedly finite) number.

What the sentence really means is that some abstract ideal "somebody" can continue counting, but that's not true either: Counting is a temporal process, and at some point everybody will be gone in a heat-dead universe. There will be no one left to count. Even long before that time, counting will be limited by the resources of the universe, which contains only a finite number of elementary particles and a finite amount of energy to increment from one integer to the next."

"Count" is a verb

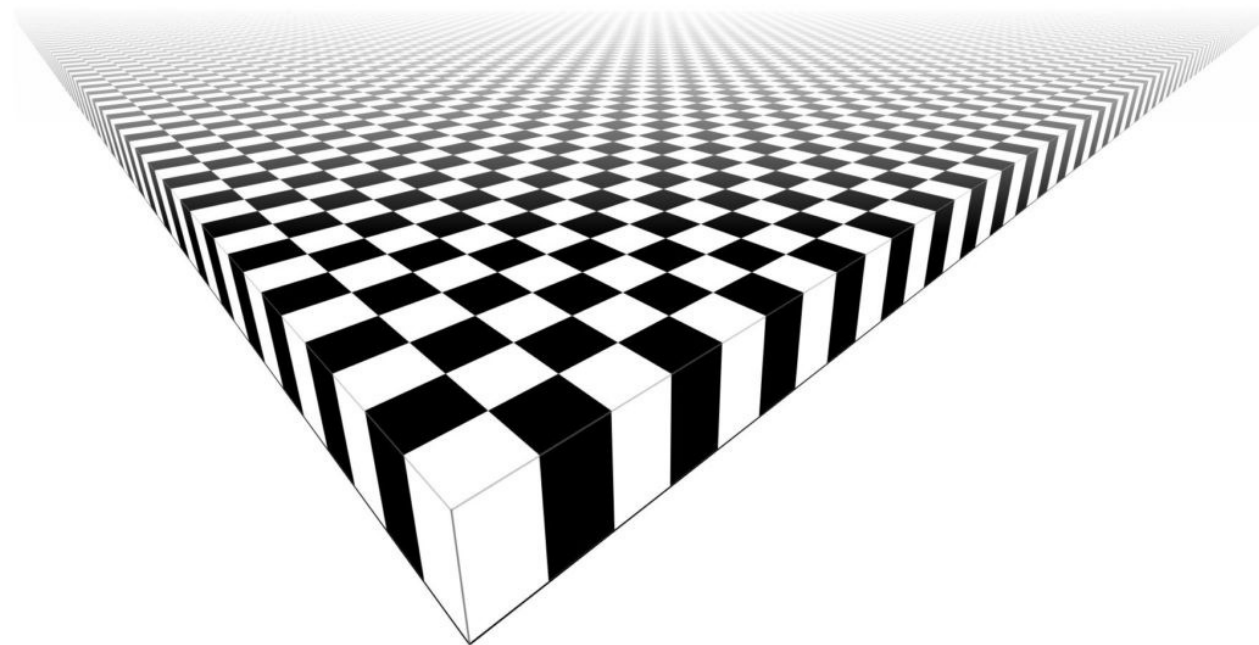
Counting "up" to infinity is the inverse of counting "down" to the ultimate subdivision of space. In both cases, the same paradox comes up. Out of curiosity, I did a bit of searching to try to find out why surreal numbers are such an interesting topic to so many people. Initially, I couldn't get much farther than seeing a trivial connection with [Zeno's paradox](https://possiblyphilosophy.wordpress.com/2008/01/09/zenos-paradox/).



(image from <https://possiblyphilosophy.wordpress.com/2008/01/09/zenos-paradox/>)

Later, I discovered a [paper on surreal numbers by Claus Tøndering](#), a Danish computer scientist who points out that he is not a mathematician by profession. Perhaps he was inspired by the true craftsmanship of the surreal number system—as an algorithmic expression with procedural elegance. He is also a Lutheran, and a believer in God—as are many people who love math, and have had a taste of the infinite.

For some reason, when reading his Introduction, I had my first, “aha” moment. The reason may simply be that I had already been marinating in the idea for a few days after reading other (more authoritative) sources. I suppose I just needed the right perspective to make it click; to open the appropriate door in my brain onto the subject.



[image by Joel David Hamkins \(http://jdh.hamkins.org/draw-an-infinite-chessboard-in-perspective/\)](http://jdh.hamkins.org/draw-an-infinite-chessboard-in-perspective/)

Subdividing the world

We all get a taste for the infinitesimal when we set our attention on the subdivision of space. In-between any two locations in space—or time, or musical pitch, or...(pick your

favorite continuum)—you can always identify another location. This of course assumes that your continuum is sufficiently smooth and your tool of measurement is sufficiently small. The knowledge that you could potentially apply the same procedure again (taking a closer look) leads to the inevitable brain-popping notion of infinitesimals.

The surreal numbers provide a novel way of describing or representing all real numbers, big and small, with infinite resolution.

That is...if you have an infinite amount of time on your hands.

Which you don't. And that is one of the core arguments in Brian Rotman's book.

Numberness

"God made the integers, all the rest is the work of man" — Leopold Kronecker



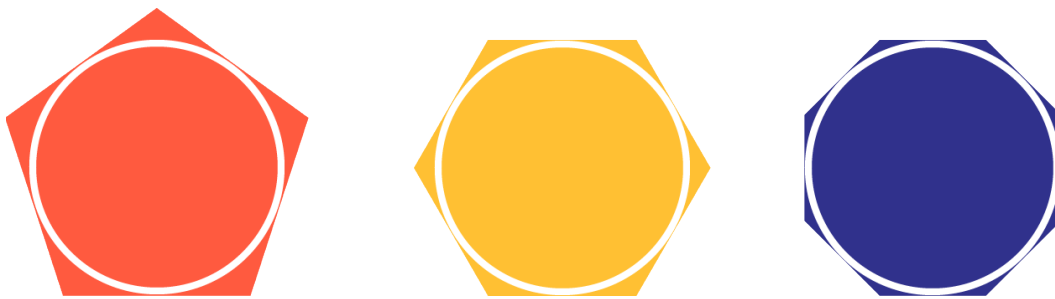
(I agree with the gist of Kroncker's famous quote, however I think it is incorrect on two accounts: (1) God did not make the integers; (2) the rest is not the work of *man*—undoubtedly some women were involved)

In response to the proposal that [Our World is an Absolute Continuum](#), I propose that our world is *unavoidably digital*...not because of physical reality per se, but because human reality runs on information, which relies on the biological *digitization* of physical reality. (See [Dennett's explanation](#) of how biological systems *discretize* analog energy into alphabetic chunks in order to generate and use information). Earth life had to evolve genes (digital information) in order to bootstrap itself into existence. Likewise, human language and culture require some form of alphabetic (digital) foundation in

order to transmit. So it seems reasonable that humans would be inclined to use digitization (numbers) to make sense of continua.

I have come to understand real numbers as not really “numbers”, but rather as number-like representations of arbitrary divisions of space (or any sufficiently-smooth continuum that we can perceive with our senses, or imagine with our minds).

Real numbers are useful when you have to carve up a continuum...and be relatively precise about it. The “numberness” of real numbers is the result of metaphorically *mating* the idea of the continuum with the idea of the natural numbers—with the help of a decimal point and some other cool tricks. Real numbers are a human-invented mutation; they can be used to represent any arbitrary partitioning of any continuum (but often with limited precision, as with the case of irrational numbers).



(image from <https://www.piday.org/pi-symbol/>)

Irrational numbers cannot be expressed using integers, or with any finite number of digits. These include (π , e , $\sqrt{2}$, ϕ). Pi is perhaps the most popular irrational number (whether or not it deserves this status). Its usefulness lies in its geometrical expression.

Its usefulness does not lie in its digits.

3.14159265358979323846264338327950288419716939937510582097494459230781640628...

Useless.

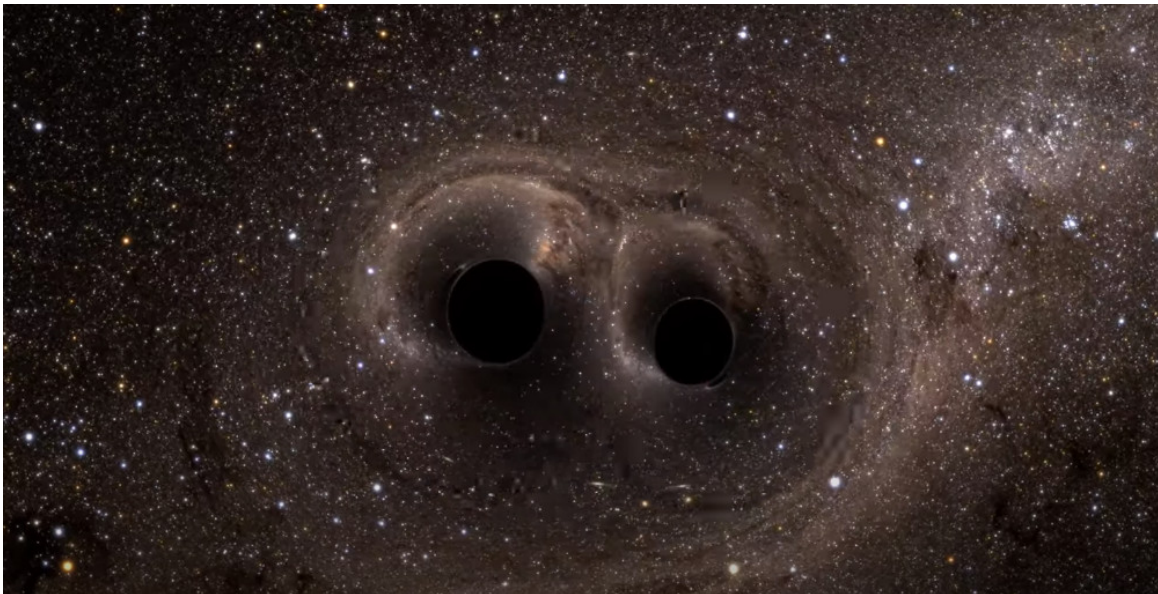
This is why any effort to [make music](#) or visual art from the digits of Pi is fruitless: the digits of Pi (while repeatable) are essentially random (and [normal](#)) and devoid of any information. And so they cannot be relied upon as the source of any [aesthetic content](#). The real number line consists *almost entirely* of irrational numbers, which are uncountably infinite. This makes [Gregory Chaitin](#) wonder:

[How real are the real numbers?](#)

Earthly origins

If real numbers are humanly-derived from integers, then what about the integers? They are more *real*, right? I say yes. Because all things are *more or less* real (i.e., some things are more *real* than others).

But even integers have a tenuous origin. Let's go back in time: there may have existed exactly 3 stars orbiting 1 black hole somewhere in a distant galaxy, long before Earth was formed. Without an intelligent observer around to take notice, can anyone claim that there were *integers* involved in this physical event?



(image from SXS project: <https://www.black-holes.org/>)

Can anyone claim that this arrangement of stellar objects proves that the integers 3 and 1 exist? (Note that any representation of the number “3”, whether printed as a character in a book or formed over time as a configuration of neurons in a brain, is not the same thing as an actual collection of three physical things.)

I propose that integers emerged within Earth’s biosphere in the form of symbol structures in evolving animal brains. Integers probably came about through [biosemiosis](#)—as a practical matter; as living things were evolving and establishing their various inner-representations of reality so as to become better at surviving. I pontificated on this in a blog post called [The Evolution of Mathematics on Planet Earth](#).

Counting makes evolutionary sense in a world of potentially-perceivable objects and events. Ask any mother crow how she feels when she comes to her nest with a worm and discovers that one of her four chicks is missing. (Yes, [crows can count](#)).



Enter humans, with their superior predictive powers:

Counting + concept of future = infinity.



(image from <https://www.pinterest.com/pin/490118371941324232/>)

Infinity is a product of human minds. It is not found in the physical world, as a physical thing, phenomenon, or attribute that can be identified or measured with repeatable scientific evidence. But it is nonetheless important to human thinking, and perhaps *essential* to human imagination. It is a product of genetic evolution and its derivative: memetic cultural evolution (memes are the cultural/informational equivalent of *genes*). Infinity is one of the most powerful memes to occupy the collective human mind-at-large. It is such a powerful meme that we collectively and unconsciously *believe* in it—even though as individuals (as rational thinking agents) we may not believe in it.

What about mathematics itself? Is it real or not? I say yes...in the same way that so many evolved meme systems are real—we humans manifest them into existence. The memetic ancestors of math had been simmering among early-human minds and human culture for a long time. And the seeds of these mathematical ideas had been growing for even longer in the minds of animals and interconnected ecosystems.



Going back even further in time, the physical structure of the universe is implicated in the emergent structures of the biosphere. So one could argue that math in fact comes

from something much deeper and more fundamental than biology. But it may not be productive to try to pinpoint the exact origin of math to any specific level of emergence or point in time. The seeds of math have been incubating over the entire spectrum, which includes the early universe, when the laws of particle physics were the the only game in town. I am proposing that math didn't truly come into being until some time after life originated on planet Earth. (The discovery of extra-terrestrial intelligence would probably require a revision of this claim, although it is likely that alien math will have some fundamental differences from ours).

Many of these ideas are derived from the philosophy of [Daniel Dennett](#), who has developed a beautiful and convincing [argument for the evolution of human minds](#)—in which both genes and memes play a key role.

So, to summarize, I propose that [math evolved in the biosphere](#). (And the biosphere is what gave birth to humans.) There was no math before there were living things, because [math is language](#), and only living things can produce language. Since language uses abstraction, the mathematical concept of infinity is able to *exist*.

[Language is, at its core, a system that is both digital and infinite.](#)

—Noam Chomsky

The infinity of human imagination

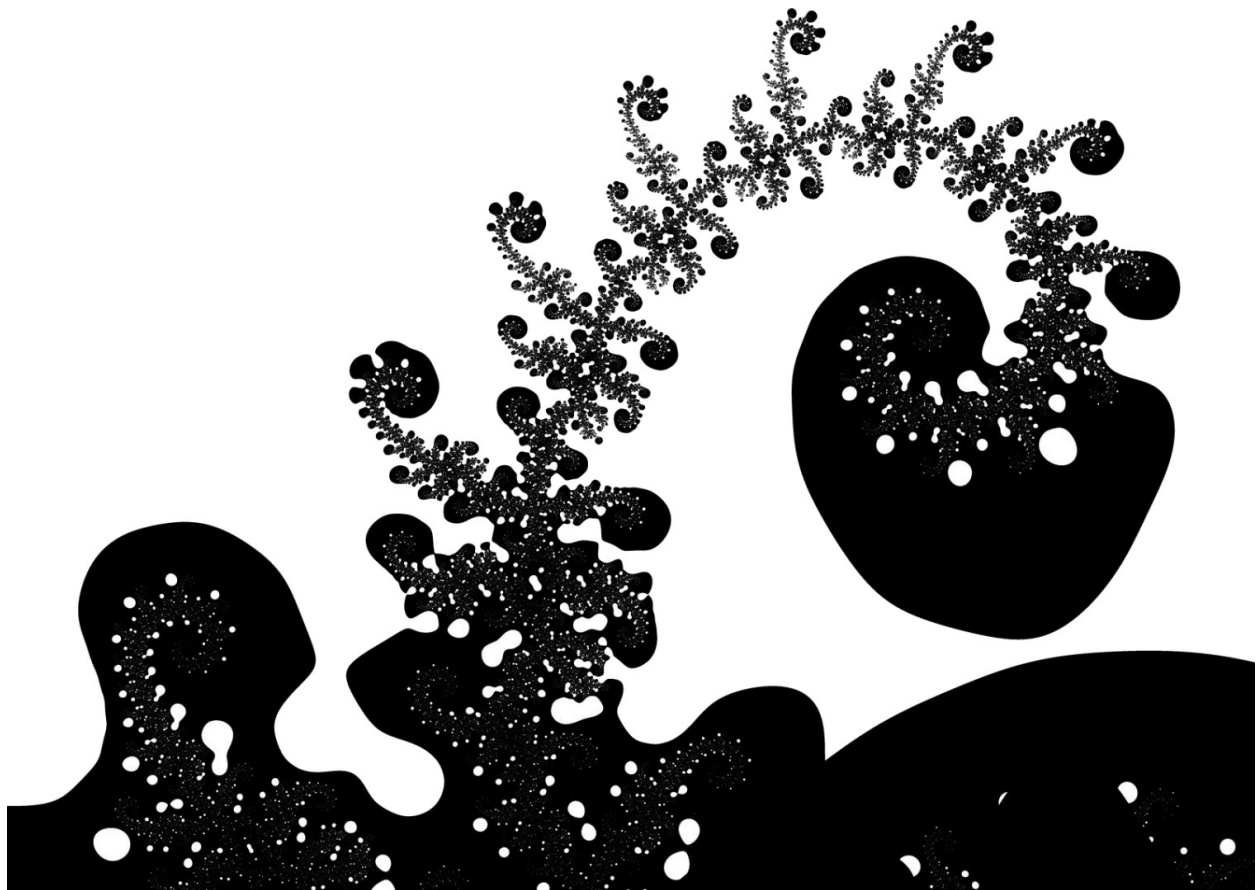


John Conway gave birth to a new way of expressing the meme of the infinite continuum, thereby paving the way for new cognitive tools, mathematical ideas, and science fiction stories.

Thanks to Conway, authors like [Rudy Rucker](#) who write about infinity now have access to a larger artistic/mathematical palette.

So, the big question is: did the surreal numbers exist before humans came along? And did Conway actually *discover* them, as he claims? I guess it depends on your point of view. But I predict no one will ever find any cosmologically-ancient fossils of surreal numbers. Or any mathematical equation for that matter.

Whether or not there is an infinite array of surreal numbers shimmering at the edge of the universe, I hope you will agree with me that human imagination is *infinite in its potential*. The greatest gift John Conway gave us was his *infinite* imagination. That's worth believing in.



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Víctor Carbajo — Fractal Domains > Storms — 24

image by Victor Carbajo (<http://www.carbajo.net/varios/fractales-i.html>)